Determinant structure of non-autonomous Toda-type integrable systems

2006 J. Phys. A: Math. Gen. 39 779

(http://iopscience.iop.org/0305-4470/39/4/004)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 779–788 doi:10.1088/0305-4470/39/4/004

Determinant structure of non-autonomous Toda-type integrable systems

Atsushi Mukaihira[1] **and Satoshi Tsujimoto**[2]

[1] Graduate School of Mathematics, Kyushu University, 6-10-1 Hakozaki, Fukuoka, 812-858, Japan
[2] Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto, 606-8501, Japan

Received 18 January 2005, in final form 31 October 2005
Published 11 January 2006
Online at stacks.iop.org/JPhysA/39/779

Abstract

The integrable chain of R_{II} type by Spiridonov–Zhedanov is studied by using the bilinear method. Bilinear equations of the system are derived by applying appropriate-dependent variable transformations. A particular solution on a semi-infinite lattice is explicitly given in terms of the Casorati-type determinants. It is shown that the R_{II} chain and the Toda-type integrable systems are connected by Bäcklund transformations.

PACS numbers: 02.30.Ik, 02.30.Gp, 05.45.Yv

1. Introduction

A class of nonlinear integrable systems are closely related to orthogonal and biorthogonal polynomials. It is well known that the Toda lattice can be derived from prescribing a spectral transformation on orthogonal polynomials. The relativistic Toda lattice is another important example, which is related to Laurent biorthogonal polynomials [4]. Such relationships between integrable systems and orthogonal, biorthogonal polynomials are now well developed based on the soliton theory.

Recently, the biorthogonal rational functions of R_I and R_{II} type were introduced by Ismail and Masson [3] in relation to the multi-point Padé approximation. These biorthogonal rational functions can be regarded as generalizations of orthogonal polynomials and Laurent biorthogonal polynomials [15]. Spiridonov and Zhedanov [11, 12] studied a spectral transformation for the R_{II} rational functions and derived a discrete integrable system, the R_{II} chain. However, there are few analyses for the R_{II} chain from the viewpoint of the integrable systems. Many fundamental properties as an integrable system are still missing and the classification of the R_{II} chain is not known. An interesting feature of the R_{II} chain is that it is the non-autonomous system: the R_{II} chain has three arbitrary parameters. It is also expected that the study of the system will reveal a feature specific to non-autonomous integrable systems. Hirota's bilinear formalism is one of the effective methods to clarify algebraic structures of integrable systems (cf [1, 2, 8]). By this method, integrable systems are transformed to bilinear equations of τ functions. For the solutions, the τ functions are expressed as determinants whose elements satisfy linear relations and the bilinear equations are reduced to identities of determinants. The τ functions reveal an underlying algebraic structures of the system and let us know relations with other integrable systems. In this sense, τ function is one of the most fundamental objects in the studies on integrable systems and we can directly approach them through the bilinear method. In this paper we study the R_{II} chain using the bilinear method.

The aim of this paper is to derive bilinear equations of the R_{II} chain and to clarify a determinant structure of a particular solution on a semi-infinite lattice. To be more precise, we transform the R_{II} chain to bilinear equations and construct a particular solution. A structure of the solution gives us an information about Bäcklund transformations which the R_{II} chain admits. Then we show that the R_{II} chain and the Toda-type integrable systems are connected by the Bäcklund transformations.

This paper is organized as follows. In section 2, we review how the R_{II} chain is derived from the R_{II} rational functions. In section 3, we derive bilinear equations of the R_{II} chain and show that a particular solution on a semi-infinite lattice is given in terms of Casorati-type determinants. In section 4, we give Bäcklund transformations for the R_{II} chain and clarify a relationship among the R_{II} chain and non-autonomous Toda-type integrable systems induced by the Bäcklund transformations. Section 5 is devoted to concluding remarks.

2. Derivation of the R_{II} chain

In this section we give a brief review of the R_{II} chain from the R_{II} rational functions. Consider polynomials $P_n(x)$ generated by the recurrence relation

$$P_{n+1}(x) + (u_n x + v_n)P_n(x) + w_n(x - \alpha_n)(x - \beta_{n-1})P_{n-1}(x) = 0, \quad n = 0, 1, \ldots \tag{1}$$

with the initial conditions

$$P_{-1}(x) = 0, \qquad P_0(x) = 1, \tag{2}$$

where u_n, v_n and w_n are some constants. The R_{II} rational functions $R_n(x)$ are defined as

$$R_n(x) = \frac{P_n(x)}{\prod_{i=1}^{n}(x - \alpha_i)}, \quad n = 1, 2, \ldots, \quad R_{-1}(x) = 0, \quad R_0(x) = 1 \tag{3}$$

under the assumptions

$$w_n \neq 0, \qquad P_n(\alpha_n) \neq 0, \quad n = 1, 2, \ldots. \tag{4}$$

These functions satisfy the recurrence relation

$$(x - \alpha_{n+1})R_{n+1}(x) + (u_n x + v_n)R_n(x) + w_n(x - \beta_{n-1})R_{n-1}(x) = 0, \quad n = 0, 1, \ldots \tag{5}$$

with the same initial conditions

$$R_{-1}(x) = 0, \qquad R_0(x) = 1. \tag{6}$$

Ismail and Masson [3] established the orthogonality relation for the R_{II} rational functions: there exists a linear functional $\mathscr{L}$ on the space of rational functions $1\big/\prod_{i=1}^{k}(x - \alpha_i)\prod_{i=1}^{l}(x - \beta_i)$, $k, l = 0, 1, \ldots$, such that the orthogonality relation

$$\mathscr{L}\left[\frac{R_n(x)}{\prod_{i=1}^{n-1}(x - \beta_i)}x^m\right] = 0, \quad m = 0, 1, \ldots, n-1 \tag{7}$$

holds. A transformation for the R_{II} rational functions is given by

$$\tilde{R}_n(x) = \frac{x-\alpha_1}{x-\lambda}\left(A_n R_{n+1}(x) + B_n R_n(x)\right), \tag{8}$$

where A_n and B_n are some constants satisfying the relation

$$A_n R_{n+1}(\lambda) + B_n R_n(\lambda) = 0. \tag{9}$$

It is easily shown that the new rational functions $\tilde{R}_n(x)$ are again the R_{II} rational functions satisfying the orthogonality relations

$$\tilde{\mathscr{L}}\left[\frac{\tilde{R}_n(x)}{\prod_{i=1}^{n-1}(x-\beta_i)}x^m\right] = 0, \qquad m = 0, 1, \ldots, n-1, \tag{10}$$

where

$$\tilde{\mathscr{L}} = \frac{x-\lambda}{x-\alpha_1}\mathscr{L}. \tag{11}$$

The transformation (8) is called the Christoffel transformation. There also exists the reciprocal transformation to the Christoffel transformation

$$R_n(x) = \frac{C_n(x-\alpha_{n+1})\tilde{R}_n(x) + D_n(x-\beta_{n-1})\tilde{R}_{n-1}(x)}{x-\alpha_1}, \tag{12}$$

where C_n and D_n are some constants. This transformation (12) is called the Geronimus transformation. These transformations (8) and (12) can be regarded as spectral transformations for the R_{II} rational functions.

The spectral transformations (8) and (12) for the R_{II} rational functions induce a discrete integrable system which the coefficients A_n, B_n, C_n and D_n satisfy. To see this, we introduce a discrete time t as the number of times that the Christoffel transformations are applied to the initial R_{II} rational functions $R_n^0(x) = R_n(x)$. Then the spectral transformations are written as follows:

$$R_n^{t+1}(x) = \frac{x-\alpha_{t+1}}{x-\lambda_t}\left(A_n^t R_{n+1}^t(x) + B_n^t R_n^t(x)\right), \tag{13a}$$

$$R_n^t(x) = \frac{C_n^t(x-\alpha_{t+n+1})R_n^{t+1}(x) + D_n^t(x-\beta_{n-1})R_{n-1}^{t+1}(x)}{x-\alpha_{t+1}}. \tag{13b}$$

From the compatibility condition of the spectral transformations (13), we derive a discrete integrable system, the R_{II} chain [11],

$$\frac{B_n^{t+1}C_n^{t+1} + A_{n-1}^{t+1}D_n^{t+1} - 1}{A_n^{t+1}C_n^{t+1}} = \frac{B_n^t C_n^t + A_n^t D_{n+1}^t - 1}{A_n^t C_{n+1}^t}, \tag{14a}$$

$$\frac{\alpha_{t+n+2}B_n^{t+1}C_n^{t+1} + \beta_{n-1}A_{n-1}^{t+1}D_n^{t+1} - \lambda_{t+1}}{A_n^{t+1}C_n^{t+1}} = \frac{\alpha_{t+n+1}B_n^t C_n^t + \beta_n A_n^t D_{n+1}^t - \lambda_t}{A_n^t C_{n+1}^t}, \tag{14b}$$

$$\frac{B_{n-1}^{t+1}D_n^{t+1}}{A_n^{t+1}C_n^{t+1}} = \frac{B_n^t D_n^t}{A_n^t C_{n+1}^t}. \tag{14c}$$

3. Determinant solution on a semi-infinite lattice

In this section we give a particular solution for the R_{II} chain by using the bilinear method.

The R_{II} chain (14) is transformed to the bilinear equations

$$(\alpha_{t+n+1}-\beta_n)\tilde{f}^t_n f^{t+1}_{n+1}+(\beta_n-\lambda_t)\tilde{f}^{t+1}_n f^t_{n+1}+(\lambda_t-\alpha_{t+n+1})g^{t+1}_n\tilde{f}^t_{n+1}=0, \tag{15a}$$

$$(\alpha_{t+n+1}-\beta_{n-1})\tilde{f}^t_n f^{t+1}_{n+1}+(\beta_{n-1}-\lambda_t)\tilde{f}^{t+1}_n f^t_{n+1}+(\lambda_t-\alpha_{t+n+1})f^{t+1}_n\tilde{g}^t_{n+1}=0, \tag{15b}$$

$$f^t_n g^{t+1}_n-g^t_n f^{t+1}_n=(\alpha_{t+n}-\lambda_t)(\beta_n-\beta_{n-1})f^{t+1}_{n-1}f^t_{n+1}, \tag{15c}$$

$$\tilde{f}^t_n\tilde{g}^{t+1}_n-\tilde{g}^t_n\tilde{f}^{t+1}_n=(\alpha_{t+n+1}-\lambda_t)(\beta_{n-1}-\beta_{n-2})\tilde{f}^{t+1}_{n-1}\tilde{f}^t_{n+1}, \tag{15d}$$

through the dependent variable transformations

$$A^t_n=\alpha_{t+n+1}-\lambda_t, \tag{16a}$$

$$B^t_n=\frac{f^t_n f^{t+1}_{n+1}}{f^{t+1}_n f^t_{n+1}}, \tag{16b}$$

$$C^t_n=\frac{f^{t+1}_n\tilde{f}^t_n}{f^t_n\tilde{f}^{t+1}_n}, \tag{16c}$$

$$D^t_n=(\alpha_{t+n+1}-\lambda_t)\frac{f^{t+1}_{n-1}\tilde{f}^t_{n+1}}{f^t_n\tilde{f}^{t+1}_n}. \tag{16d}$$

It can be shown that if f^t_n, $\tilde{f}^t_n$, g^t_n and $\tilde{g}^t_n$ satisfy the bilinear equations (15), then A^t_n, B^t_n, C^t_n and D^t_n satisfy the R_{II} chain (14).

We give a solution for the R_{II} chain on a semi-infinite lattice.

Theorem 1. *Define the τ functions $\tau^{k,l,t}_n$ and $\sigma^{k,l,t}_n$ as*

$$\tau^{k,l,t}_n=\begin{vmatrix} c^t_{k,l}(s) & c^t_{k,l+1}(s) & \cdots & c^t_{k,l+n-1}(s)\\ c^t_{k+1,l}(s) & c^t_{k+1,l+1}(s) & \cdots & c^t_{k+1,l+n-1}(s)\\ \vdots & \vdots & & \vdots\\ c^t_{k+n-1,l}(s) & c^t_{k+n-1,l+1}(s) & \cdots & c^t_{k+n-1,l+n-1}(s)\end{vmatrix}, \tag{17a}$$

$$\sigma^{k,l,t}_n=\begin{vmatrix} c^t_{k,l}(s) & \cdots & c^t_{k,l+n-2}(s) & d^t_{k,l+n-1}(s)\\ c^t_{k+1,l}(s) & \cdots & c^t_{k+1,l+n-2}(s) & d^t_{k+1,l+n-1}(s)\\ \vdots & & \vdots & \vdots\\ c^t_{k+n-1,l}(s) & \cdots & c^t_{k+n-1,l+n-2}(s) & d^t_{k+n-1,l+n-1}(s)\end{vmatrix}, \tag{17b}$$

where the elements $c^t_{k,l}(s)$ and $d^t_{k,l}(s)$ satisfy the dispersion relations

$$c^t_{k,l}(s+1)=c^t_{k-1,l}(s)+\alpha_{k+t}c^t_{k,l}(s)=c^t_{k,l-1}(s)+\beta_l c^t_{k,l}(s)=c^{t+1}_{k-1,l}(s)+\lambda_t c^t_{k,l}(s), \tag{18a}$$

$$d^t_{k,l}(s)=c^t_{k,l+1}(s+1)-\beta_l c^t_{k,l+1}(s). \tag{18b}$$

Then the τ functions (17) give a solution for the bilinear equations

$$(\alpha_{k+t+n+1}-\beta_{l+n})\tau^{k+1,l-1,t}_n\tau^{k,l,t+1}_{n+1}+(\beta_{l+n}-\lambda_t)\tau^{k+1,l-1,t+1}_n\tau^{k,l,t}_{n+1}$$
$$+(\lambda_t-\alpha_{k+t+n+1})\sigma^{k,l,t+1}_n\tau^{k+1,l-1,t}_{n+1}=0, \tag{19a}$$

$$(\alpha_{k+t+n+1}-\beta_{l+n-1})\tau^{k+1,l-1,t}_n\tau^{k,l,t+1}_{n+1}+(\beta_{l+n-1}-\lambda_t)\tau^{k+1,l-1,t+1}_n\tau^{k,l,t}_{n+1}$$
$$+(\lambda_t-\alpha_{k+t+n+1})\tau^{k,l,t+1}_n\sigma^{k+1,l-1,t}_{n+1}=0, \tag{19b}$$

$$\tau_n^{k,l,t}\sigma_n^{k,l,t+1}-\sigma_n^{k,l,t}\tau_n^{k,l,t+1}=(\alpha_{k+t+n}-\lambda_t)(\beta_{l+n}-\beta_{l+n-1})\tau_{n-1}^{k,l,t+1}\tau_{n+1}^{k,l,t}, \tag{19c}$$

on the semi-infinite lattice

$$\tau_{-1}^{k,l,t}=\tau_{-2}^{k,l,t}=\cdots=0, \qquad \sigma_{-1}^{k,l,t}=\sigma_{-2}^{k,l,t}=\cdots=0, \qquad \tau_0^{k,l,t}=\sigma_0^{k,l,t}=1. \tag{20}$$

An example of the function $c_{k,l}^t(s)$ is given by

$$c_{k,l}^t(s)=\sum_{j=1}^{\infty}\frac{w_jx_j^s\prod_{i=-\infty}^{t-1}(x_j-\lambda_i)}{\prod_{i=-\infty}^{t+k}(x_j-\alpha_i)\prod_{i=-\infty}^{l}(x_i-\beta_i)}. \tag{21}$$

As the bilinear equations (19) are reduced to those of the R_{II} chain (15) in the case of $k=l=0$, we have obtained a solution for the R_{II} chain.

Corollary 2. *For the τ functions (17),*

$$f_n^t=\tau_n^{0,0,t}, \qquad \tilde{f}_n^t=\tau_n^{1,-1,t}, \qquad g_n^t=\sigma_n^{0,0,t}, \qquad \tilde{g}_n^t=\sigma_n^{1,-1,t}, \tag{22}$$

give a solution for the bilinear equations of the R_{II} chain (15) on the semi-infinite lattice (20).

We note that the function $\sigma_n^{1,-1,t}$ is the auxiliary variable which is introduced in decoupling the R_{II} chain into bilinear equations. In terms of the variables B_n^t, C_n^t and D_n^t, the corresponding boundary condition is given by

$$B_{-1}^t=B_{-2}^t=\cdots=1, \qquad C_0^t=C_{-1}^t=\cdots=1, \qquad D_0^t=D_{-1}^t=\cdots=0. \tag{23}$$

Proof of Theorem 1. We show that the τ functions (17) satisfy the bilinear equations (15).

Consider the identity

$$\begin{vmatrix} f_1 & \cdots & f_n & a_1 & & \emptyset & & a_2 & a_3 \\ & \emptyset & & a_1 & f_1 & \cdots & f_{n-1} & a_2 & a_3 \end{vmatrix}=0, \tag{24}$$

where f_i, a_i are arbitrary $(n+1)$-dimensional column vectors. Applying the Laplace expansion to the left-hand side of the identity (24), we obtain

$$\begin{aligned} &|f_1 \ \cdots \ f_{n-1} \ f_n \ a_1|\cdot|f_1 \ \cdots \ f_{n-1} \ a_2 \ a_3| \\ &-|f_1 \ \cdots \ f_{n-1} \ f_n \ a_2|\cdot|f_1 \ \cdots \ f_{n-1} \ a_1 \ a_3| \\ &+|f_1 \ \cdots \ f_{n-1} \ f_n \ a_3|\cdot|f_1 \ \cdots \ f_{n-1} \ a_1 \ a_2|=0, \end{aligned} \tag{25}$$

which is one of the Plücker relations. The bilinear equation (15a) follows from the Plücker relation (25) with

$$f_i=\left(c_{k+i,l+n-2}^{t+1}(s+n-1) \ \cdots \ c_{k+i,l+n-2}^{t+1}(s) \ \ c_{k+i,l+n-1}^{t+1}(s)\right)^\top, \tag{26a}$$

$$a_1=\left(d_{k+n-1,l+n-1}^{t+1}(s+n-1) \ \cdots \ d_{k+n-1,l+n-1}^{t+1}(s) \ \ c_{k+n-1,l+n}^{t+1}(s)\right)^\top, \tag{26b}$$

$$a_2=\left(c_{k+n,l+n-2}^{t}(s+n-1) \ \cdots \ c_{k+n,l+n-2}^{t}(s) \ \ c_{k+n,l+n-1}^{t}(s)\right)^\top, \tag{26c}$$

$$a_3=\left(0 \ \cdots \ 0 \ \ 1\right)^\top. \tag{26d}$$

Indeed, we can see that

$$|f_1 \ \cdots \ f_{n-1} \ f_n \ a_1|=-(\alpha_{k+t+n+1}-\beta_{l+n})\tau_{n+1}^{k,l,t+1}, \tag{27a}$$

$$|f_1 \ \cdots \ f_{n-1} \ a_2 \ a_3|=\tau_n^{k+1,l-1,t}, \tag{27b}$$

$$|f_1 \ \cdots \ f_{n-1} \ f_n \ a_2|=(\lambda_t-\alpha_{k+t+n+1})\tau_{n+1}^{k+1,l-1,t}, \tag{27c}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad a_1 \quad a_3| = \sigma_n^{k,l,t+1}, \tag{27d}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad f_n \quad a_3| = \tau_n^{k+1,l-1,t+1}, \tag{27e}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad a_1 \quad a_2| = -(\beta_{l+n} - \lambda_t)\tau_{n+1}^{k,l,t}. \tag{27f}$$

Similarly, the bilinear equation (15b) is reduced to the Plücker relation (25) with

$$f_i = \left(c_{k+i,l+n-2}^{t+1}(s+n-1) \quad \cdots \quad c_{k+i,l+n-2}^{t+1}(s) \quad d_{k+i,l+n-1}^{t+1}(s)\right)^{\top}, \tag{28a}$$

$$a_1 = \left(c_{k+n-1,l+n-1}^{t+1}(s+n-1) \quad \cdots \quad c_{k+n-1,l+n-1}^{t+1}(s) \quad c_{k+n-1,l+n}^{t+1}(s)\right)^{\top}, \tag{28b}$$

$$a_2 = \left(c_{k+n,l+n-2}^{t}(s+n-1) \quad \cdots \quad c_{k+n,l+n-2}^{t}(s) \quad c_{k+n,l+n-1}^{t}(s)\right)^{\top}, \tag{28c}$$

$$a_3 = \left(0 \quad \cdots \quad 0 \quad 1\right)^{\top}. \tag{28d}$$

Let D be some determinant, and $D\left[\begin{smallmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{smallmatrix}\right]$ be the determinant with the $i_1, \ldots, i_n$th rows and the $j_1, \ldots, j_n$th columns removed from D. Then the following identity is satisfied:

$$D \cdot D\begin{bmatrix} i & k \\ j & l \end{bmatrix} = D\begin{bmatrix} i \\ j \end{bmatrix} D\begin{bmatrix} k \\ l \end{bmatrix} - D\begin{bmatrix} i \\ l \end{bmatrix} D\begin{bmatrix} k \\ j \end{bmatrix}, \tag{29}$$

which is called the Jacobi identity. The bilinear equation (15c) follows from the Jacobi identity (29) with $i = j = 1$, $k = l = n + 1$, where D is given by

$$D = \begin{vmatrix} d_{k+n-1,l+n-1}^{t}(s) & c_{k+n-1,l}^{t}(s) & \cdots & c_{k+n-1,l+n-1}^{t}(s) \\ d_{k,l+n-1}^{t+1}(s) & c_{k,l}^{t+1}(s) & \cdots & c_{k,l+n-1}^{t+1}(s) \\ \vdots & \vdots & & \vdots \\ d_{k+n-1,l+n-1}^{t+1}(s) & c_{k+n-1,l}^{t+1}(s) & \cdots & c_{k+n-1,l+n-1}^{t+1}(s) \end{vmatrix}. \tag{30}$$

Indeed, we can see that

$$D = -(\alpha_{k+t+n} - \lambda_t)(\beta_{l+n} - \beta_{l+n-1})\tau_{n+1}^{k,l,t}, \tag{31a}$$

$$D\begin{bmatrix} 1 & n+1 \\ 1 & n+1 \end{bmatrix} = \tau_{n-1}^{k,l,t+1}, \tag{31b}$$

$$D\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tau_n^{k,l,t+1}, \tag{31c}$$

$$D\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} = \sigma_n^{k,l,t}, \tag{31d}$$

$$D\begin{bmatrix} 1 \\ n+1 \end{bmatrix} = (-1)^{n-1}\sigma_n^{k,l,t+1}, \tag{31e}$$

$$D\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = (-1)^{n-1}\tau_n^{k,l,t}. \tag{31f}$$

This completes the proof. □

4. Bäcklund transformations for the non-autonomous Toda-type integrable systems

The R_{II} chain has close relations with many other Toda-type integrable systems. In this section we discuss Bäcklund transformations of Toda-type integrable systems including the R_{II} chain.

The Bäcklund transformation is a transformation which maps some solution to other one and induces a discrete integrable system. A typical example is the discrete Toda lattice, which is nothing but the Bäcklund transformation for the continuous Toda lattice. The τ functions (17) defined in the previous section depend on the variables n, k, l and t. For the R_{II} chain, the variables k and l are auxiliary variables and describe the Bäcklund transformations. We first give Bäcklund transformations for the R_{II} chain in terms of bilinear equations. The Bäcklund transformations induce many non-autonomous Toda-type integrable systems such as the Toda lattice, the generalized relativistic Toda lattice, the R_I chain and others. We then clarify a relationship among the R_{II} chain and those Toda-type integrable systems.

We introduce the independent variable z as follows:

$$c_{k,l}^t(s+1) = \frac{\mathrm{d}}{\mathrm{d}z} c_{k,l}^t(s) = c_{k-1,l}^t(s) + \alpha_{k+t} c_{k,l}^t(s) = c_{k,l-1}^t(s) + \beta_l c_{k,l}^t(s)$$
$$= c_{k-1,l}^{t+1}(s) + \lambda_t c_{k,l}^t(s). \tag{32}$$

The variable z can be regarded as a continuous variable corresponds to the discrete variable k or l. We first propose Bäcklund transformations for the R_{II} chain in terms of bilinear equations.

Proposition 3. *The τ functions (17) satisfy the bilinear equations*

$$\tau_n^{k,l,t}\tau_n^{k+1,l+1,t} - \tau_n^{k,l+1,t}\tau_n^{k+1,l,t} = \tau_{n-1}^{k+1,l+1,t}\tau_{n+1}^{k,l,t}, \tag{33a}$$
$$\tau_n^{k+1,l,t}\tau_{n+1}^{k-1,l,t} + (\alpha_{k+t+n} - \beta_{l+n})\tau_n^{k,l,t}\tau_{n+1}^{k,l,t} = \tau_n^{k,l+1,t}\tau_{n+1}^{k,l-1,t}, \tag{33b}$$
$$\tau_n^{k,l+1,t}\tau_n^{k,l-1,t} - \tau_{n-1}^{k+1,l,t}\tau_{n+1}^{k-1,l,t} = \tau_n^{k,l,t}\sigma_n^{k,l,t}, \tag{33c}$$
$$\tau_n^{k+1,l+1,t}\sigma_{n+1}^{k,l,t} - \sigma_n^{k+1,l+1,t}\tau_{n+1}^{k,l,t} = (\beta_{l+n+1} - \beta_{l+n})\tau_n^{k+1,l,t}\tau_{n+1}^{k,l+1,t}, \tag{33d}$$
$$D_z\tau_n^{k,l,t}\cdot\tau_n^{k,l+1,t} = \tau_{n-1}^{k+1,l+1,t}\tau_{n+1}^{k-1,l,t}, \tag{33e}$$
$$(D_z + \beta_{l+n})\tau_n^{k+1,l,t}\cdot\tau_{n+1}^{k,l,t} = -\tau_n^{k+1,l+1,t}\tau_{n+1}^{k,l-1,t}, \tag{33f}$$
$$D_z^2\tau_n^{k,l,t}\cdot\tau_n^{k,l,t} = 2\tau_{n-1}^{k,l,t}\tau_{n+1}^{k,l,t}, \tag{33g}$$

where D_z is the bilinear differential operator defined by

$$D_z^n f(z)\cdot g(z) = \frac{\mathrm{d}^n}{\mathrm{d}z'^n} f(z+z')g(z-z')|_{z'=0}. \tag{34}$$

Many significant Toda-type integrable systems are derived from a part of the bilinear equations (19) and (33) satisfied with the τ functions (17). Thus the rest of the bilinear equations are the Bäcklund transformations for those integrable systems. In what follows we list derived nonlinear equations and its Bäcklund transformations:

(i) Toda lattice

The bilinear equation (33g) is transformed to the Toda lattice

$$\frac{\mathrm{d}a_n}{\mathrm{d}z} = a_n(b_n - b_{n-1}), \tag{35a}$$
$$\frac{\mathrm{d}b_n}{\mathrm{d}z} = a_{n+1} - a_n \tag{35b}$$

through the dependent variable transformations

$$a_n = \frac{\tau_{n-1}^{k,l,t}\tau_{n+1}^{k,l,t}}{\left(\tau_n^{k,l,t}\right)^2}, \tag{36a}$$

$$b_n = \frac{\mathrm{d}\log\left(\tau_{n+1}^{k,l,t}/\tau_n^{k,l,t}\right)}{\mathrm{d}z}. \tag{36b}$$

The bilinear equations (19) and (33*a*)–(33*f*) are the Bäcklund transformations for the Toda lattice. This system has no arbitrary parameter.

(ii) Discrete Toda lattice

The bilinear equations (33*c*) and (33*d*) are transformed to the discrete Toda lattice

$$A_n^{l-1} + B_n^{l-1} + \beta_{l-1} = A_n^l + B_{n+1}^l + \beta_l, \tag{37a}$$

$$A_{n-1}^{l-1} B_n^{l-1} = A_n^l B_n^l \tag{37b}$$

through the dependent variable transformation

$$A_n^l = \frac{\tau_n^{k-n+1,l-n+1,t}\tau_{n+1}^{k-n,l-n-1,t}}{\tau_n^{k-n+1,l-n,t}\tau_{n+1}^{k-n,l-n,t}}, \tag{38a}$$

$$B_n^l = -\frac{\tau_{n-1}^{k-n+2,l-n+1,t}\tau_{n+1}^{k-n,l-n,t}}{\tau_n^{k-n+1,l-n+1,t}\tau_n^{k-n+1,l-n,t}}. \tag{38b}$$

It is well known that the continuous and discrete Toda lattice are induced by spectral transformations for orthogonal polynomials. The bilinear equations (19) and (33*a*), (33*b*), (33*e*)–(33*g*) are the Bäcklund transformations for the discrete Toda lattice. This system has one arbitrary parameter β_l.

(iii) Generalized relativistic Toda lattice

Through the dependent variable transformations

$$a_n = -\frac{\tau_{n-1}^{k-n+2,l-1,t}\tau_{n+1}^{k-n,l,t}}{\tau_n^{k-n+1,l,t}\tau_n^{k-n+1,l-1,t}}, \tag{39a}$$

$$b_n = \frac{\tau_n^{k-n+1,l-1,t}\sigma_{n+1}^{k-n,l,t}}{\tau_n^{k-n+1,l,t}\tau_{n+1}^{k-n,l-1,t}} + \beta_{l+n}, \tag{39b}$$

the bilinear equations (33*c*)–(33*f*) are transformed to

$$\frac{\mathrm{d}a_n}{\mathrm{d}z} = a_n(a_{n-1} - a_{n+1} + b_n - b_{n-1}), \tag{3a}$$

$$\frac{\mathrm{d}b_n}{\mathrm{d}z} = \beta_{l+n-1}a_{n+1} - \beta_{l+n}a_n + b_n(a_n - a_{n+1}), \tag{3b}$$

which is the non-autonomous generalization of the relativistic Toda lattice. The generalized relativistic Toda lattice was derived by Vinet and Zhedanov [14] in the study of a spectral transformation for the R_I rational functions. The bilinear equations (19) and (33*a*), (33*b*), (33*g*) are the Bäcklund transformations for the generalized relativistic Toda lattice. This system has one arbitrary parameter β_{l+n} and is reduced to the ordinary relativistic Toda lattice [10, 9] by specializing β_{l+n} to a constant.

(iv) R_I chain

The bilinear equations (33*a*)–(33*d*) are transformed to the R_I chain,

$$\frac{A_{n-1}^{l-1}C_n^{l-1} - 1}{A_n^{l-1}} = \frac{A_n^l C_{n+1}^l - 1}{A_n^l}, \tag{4a}$$

$$\frac{\alpha_{k+t+n}A_{n-1}^{l-1}C_n^{l-1}-B_n^{l-1}-\beta_{l-1}}{A_n^{l-1}}=\frac{\alpha_{k+t+n+1}A_n^lC_{n+1}^l-B_n^l-\beta_l}{A_n^l}, \tag{4b}$$

$$\frac{B_{n-1}^{l-1}C_n^{l-1}}{A_n^{l-1}}=\frac{B_n^lC_n^l}{A_n^l}, \tag{4c}$$

through the dependent variable transformations

$$A_n^l=-\frac{\tau_n^{k-n+2,l-1,t}\tau_{n+1}^{k-n,l,t}}{\tau_n^{k-n+1,l-1,t}\tau_{n+1}^{k-n+1,l,t}}, \tag{5a}$$

$$B_n^l=\frac{\tau_n^{k-n+1,l,t}\tau_{n+1}^{k-n+1,l-1,t}}{\tau_n^{k-n+1,l-1,t}\tau_{n+1}^{k-n+1,l,t}}, \tag{5b}$$

$$C_n^l=-\frac{\tau_{n-1}^{k-n+2,l-1,t}\tau_{n+1}^{k-n,l,t}}{\tau_n^{k-n+1,l,t}\tau_n^{k-n+1,l-1,t}}. \tag{5c}$$

The R_I chain was also derived through the spectral transformation for the R_I rational functions. A determinant solution for the system was discussed in [7]. The bilinear equations (19) and (33*e*)–(33*g*) are the Bäcklund transformations for the R_I chain. This system has two arbitrary parameters α_{k+t+n} and β_l and is reduced to the discrete relativistic Toda lattice [13, 5] by specializing the arbitrary parameter α_{k+t+n} to a constant.

(v) R_{II} chain

The bilinear equations (19) are transformed to the R_{II} chain,

$$\frac{B_n^{t+1}C_n^{t+1}+A_{n-1}^{t+1}D_n^{t+1}-1}{A_n^{t+1}C_n^{t+1}}=\frac{B_n^tC_n^t+A_n^tD_{n+1}^t-1}{A_n^tC_{n+1}^t}, \tag{6a}$$

$$\frac{\alpha_{k+t+n+2}B_n^{t+1}C_n^{t+1}+\beta_{l+n-1}A_{n-1}^{t+1}D_n^{t+1}-\lambda_{t+1}}{A_n^{t+1}C_n^{t+1}} \tag{6b}$$

$$=\frac{\alpha_{k+t+n+1}B_n^tC_n^t+\beta_{l+n}A_n^tD_{n+1}^t-\lambda_t}{A_n^tC_{n+1}^t}, \tag{6c}$$

$$\frac{B_{n-1}^{t+1}D_n^{t+1}}{A_n^{t+1}C_n^{t+1}}=\frac{B_n^tD_n^t}{A_n^tC_{n+1}^t}, \tag{6d}$$

through the dependent variable transformations

$$A_n^{k,l,t}=\alpha_{k+t+n+1}-\lambda_t, \tag{7a}$$

$$B_n^{k,l,t}=\frac{\tau_n^{k,l,t}\tau_{n+1}^{k,l,t+1}}{\tau_n^{k,l,t+1}\tau_{n+1}^{k,l,t}}, \tag{7b}$$

$$C_n^{k,l,t}=\frac{\tau_n^{k,l,t+1}\tau_n^{k+1,l-1,t}}{\tau_n^{k,l,t}\tau_n^{k+1,l-1,t+1}}, \tag{7c}$$

$$D_n^{k,l,t}=(\alpha_{k+t+n+1}-\lambda_t)\frac{\tau_{n-1}^{k,l,t+1}\tau_{n+1}^{k+1,l-1,t}}{\tau_n^{k,l,t}\tau_n^{k+1,l-1,t+1}}. \tag{7d}$$

The bilinear equations (33) are the Bäcklund transformations for the R_{II} chain. This system has three arbitrary parameters: α_{k+t+n}, β_{l+n} and λ_t.

5. Concluding remarks

In this paper, we have derived bilinear equations of the R_{II} chain and shown that a particular solution on a semi-infinite lattice is given in terms of Casorati-type determinants. We have also given Bäcklund transformations for the R_{II} chain in terms of the bilinear equation and clarified a relationship among the R_{II} chain and non-autonomous Toda-type integrable systems induced by the Bäcklund transformations.

The τ functions in the particular solution which we constructed depend on the variables n, k, l and t. Among them, n and t denote the independent variables of the R_{II} chain. The variables k and l which do not appear in the equations describe the Bäcklund transformations of the R_{II} chain. The three non-autonomous parameters of the R_{II} chain reflect such a structure of the system. As shown in section 4, the Bäcklund transformations induce many significant non-autonomous Toda-type integrable systems. The τ functions of the R_{II} chain give us a unified approach to these non-autonomous Toda-type integrable systems including the R_{II} chain.

We have considered a solution for the R_{II} chain on a semi-infinite lattice in this paper. Other particular solutions on various types of lattices such as an infinite lattice and a periodic lattice are a future subject to be studied.

Acknowledgments

The authors are grateful to Professor Yoshimasa Nakamura and Professor Alexei Zhedanov for many fruitful discussions and helpful advice. One of the authors (AM) thanks Professor Reinout Quispel for hospitality during his stay at La Trobe University. This paper is supported in part by Grant-in-Aid for Scientific Research (nos 15005102 and 15540119) from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

References

[1] Hirota R 1977 Nonlinear partial difference equations: II. Discrete-time Toda equation *J. Phys. Soc. Japan* **43** 2074–8
[2] Hirota R 1987 Discrete two-dimensional Toda molecule equation *J. Phys. Soc. Japan* **56** 4285–8
[3] Ismail M E H and Masson D R 1995 Generalized orthogonality and continued fractions *J. Approx. Theory* **83** 1–40
[4] Kharchev S, Mironov A and Zhedanov A 1997 Faces of relativistic Toda chain *Int. J. Mod. Phys.* A **12** 2675–724
[5] Maruno K, Kajiwara K and Oikawa M 1998 Casorati determinant solution for the discrete-time relativistic Toda lattice equation *Phys. Lett.* A **241** 335–43
[6] Minesaki Y and Nakamura Y 2001 The discrete relativistic Toda molecule equation and a Padé approximation algorithm *Numer. Algorithms* **27** 219–35
[7] Mukaihira A and Tsujimoto S 2004 Determinant structure of R_I type discrete integrable system *J. Phys. A: Math. Gen.* **37** 4557–65
[8] Ohta Y, Hirota R, Tsujimoto S and Imai T 1993 Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy *J. Phys. Soc. Japan* **62** 1872–86
[9] Ohta Y, Kajiwara K, Matsukidaira J and Satsuma J 1993 Casorati determinant solution for the relativistic Toda lattice equation *J. Math. Phys.* **34** 5190–204
[10] Ruijsenaars S N M 1990 Relativistic Toda systems *Commun. Math. Phys.* **133** 217–47
[11] Spiridonov V and Zhedanov A 2000 Spectral transformation chains and some new biorthogonal rational functions *Commun. Math. Phys.* **210** 49–83
[12] Spiridonov V and Zhedanov A 2003 To the theory of biorthogonal rational functions *RIMS Kokyuroku* **1302** 172–92
[13] Suris Yu B 1996 A discrete-time relativistic Toda lattice *J. Phys. A: Math. Gen.* **29** 451–65
[14] Vinet L and Zhedanov A 1998 An integrable chain and bi-orthogonal polynomials *Lett. Math. Phys.* **46** 233–45
[15] Zhedanov A 1999 Biorthogonal rational functions and generalized eigenvalue problem *J. Approx. Theory* **101** 303–29